

THE UNIVERSITY OF CHICAGO

THE MINIMA OF INDEFINITE QUATERNARY
QUADRATIC FORMS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF MATHEMATICS

BY

ALEXANDER OPPENHEIM

PRIVATE EDITION

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THE MINIMA OF INDEFINITE QUATERNARY QUADRATIC FORMS.¹

BY ALEXANDER OPPENHEIM.

1. **Introduction.** Let $f = \sum a_{ik}x_ix_k$ ($a_{ik} = a_{ki}$) be a real quadratic form with real coefficients in n variables $x_1, x_2, \dots, x_n$. The variables assume integral values only, the set $0, 0, \dots, 0$ excepted. One of the many problems associated with quadratic forms is that of finding the upper bound of $L(f)$, the absolute lower bound of f , when f runs over the set of forms which have a given hessian $a = \|a_{ik}\|$, a not zero.²

If f is a positive definite binary form, it is well known that $L^2 \leq 4a/3$, and that, if equality holds, f is equivalent to the form $L(x^2 + xy + y^2)$. Further we can find a form f of hessian a with $4a/3 > L^2 > 4a/3 - e$, for any positive e , however small.

In sharp contradistinction to the last result is the behavior of indefinite forms. Thus, if f is an indefinite binary quadratic form, then $L^2 \leq -4a/5$. Equality implies that $f \sim L(x^2 + xy - y^2)$. If equality does not hold, then necessarily $L^2 \leq -\frac{1}{2}a$, a much stronger inequality. Equality here implies $f \sim L(x^2 + 2xy - y^2)$. And so on. The complete theorem is due to Markoff.³ It is Theorem 1 in § 2.

Similar results hold for indefinite ternary and quaternary quadratic forms. The first three minima for the former were obtained by Markoff⁴ for forms with commensurable coefficients, and extended to the fourth minimum by Dickson⁵ for forms which attain their lower bound. The extension to forms which do not necessarily attain their lower bound was made by the writer.⁶ The results are given in Theorem 2 in § 3.

In the case of quaternary forms, the problem splits up naturally into two parts, for indefinite quaternary forms can have signature zero or ± 2 . One difference between quaternary forms on the one hand and ternary and binary forms on the other hand may be noted. In the latter case corresponding, say, to $L^2 = -\frac{1}{2}a$, we have just one class of indefinite binary forms with this value of L . For indefinite quaternary forms of

¹Received May 26, 1930.—Doctoral Dissertation, University of Chicago, 1930.

²If a is zero, it is easy to see that $L = 0$.

³A. Markoff, 1, 2. For an exposition see L. E. Dickson, 2.

⁴Markoff, 3; exposition in Dickson, 2.

⁵Dickson, 2.

⁶See Dickson, 2.

signature ± 2 with $L^4 = -\frac{4}{15}a$, on the contrary, we obtain *two non-equivalent* classes of forms (Theorem B below).

My results are contained in two theorems which follow.

THEOREM A. *Let f be an indefinite quaternary quadratic form of signature zero, so that $a > 0$. Let $L^4(f) > \frac{64}{875}a$. Then necessarily⁷*

$$(A1) \quad L^4 = \frac{4}{9}a \quad \text{or} \quad L^4 = \frac{4}{17}a,$$

and $\pm f/L$ is equivalent to

$$(A2) \quad \begin{array}{l} x^2 + xt + t^2 - 2y^2 - 2yz - 2z^2 \quad \text{or} \\ x^2 - y^2 - z^2 + t^2 + xt + yt + 2zt + 3zx + 2xy + yz \end{array}$$

respectively.

THEOREM B. *Let f be an indefinite quaternary quadratic form of signature ± 2 , so that $a < 0$. Let $L^4(f) \geq -\frac{1}{5}a$. Then necessarily*

$$(B1) \quad -L^4/a = \frac{4}{7} \quad \text{or} \quad \frac{4}{15} \quad \text{or} \quad \frac{2}{9}.$$

In the first case, $\pm f/L$ is equivalent to the form

$$(B2) \quad t^2 - x^2 - y^2 - z^2 + xt + yt + zt, \quad -L^4/a = \frac{4}{7}.$$

In the second case, $\pm f/L$ is equivalent to one of the forms

$$(B3) \quad x^2 + xt - t^2 + 2(y^2 + yz + z^2), \quad 2(x^2 + xt - t^2) + y^2 + yz + z^2, \quad -L^4/a = \frac{4}{15}.$$

In the third case, $\pm f/L$ is equivalent to the form

$$(B4) \quad x^2 + xy + y^2 - 6z^2 + t^2, \quad -L^4/a = \frac{2}{9}.$$

We have immediately the

COROLLARY. *Let f be an indefinite quaternary quadratic form. If f does not represent both L and $-L$ (in particular, if L is not attained), or if the coefficients of f are incommensurable, then $L^4 < \frac{1}{5}|a|$.*

In an earlier (unpublished) paper I obtained the first minima and forms in Theorems A and B. An account of this paper appears in Dickson, 2, Ch. IX. To this treatise and to Dickson, 1, I refer for expositions of the results given in §§ 2 and 3 on indefinite binary and ternary quadratic forms.

In the case of indefinite forms in five or more variables, no such theorems hold, for it is known that all indefinite quadratic forms in five or more variables with *commensurable* coefficients are *null*, so that $L(f)$ is necessarily zero. Very likely $L(f)$ is zero also when the coefficients of f are *incommensurable*, but as yet this has not been proved.

⁷ Another deduction of (A 1.1) is given in Oppenheim, 1.

Method of proof. Since L is the absolute lower bound of f , we have $|f| \geq L$ for all sets of values of $x_1, x_2, \dots, x_n$ other than $0, 0, \dots, 0$. By means of this condition, we can derive various inequalities between the coefficients of f and so determine a relation between L and a . Since a and L are homogeneous of degrees n and 1 respectively in the coefficients of f , such a relation, it is clear, will be of the form $L^n \leq K(n, s) |a|$ where K is a number depending only on n and s , the signature of the set of forms under consideration.

In practice it is better to take linear combinations of the variables and so obtain binary or ternary forms, which are *sections* of f . Evidently any such section, g , will have a lower bound $\geq L$, since every value of g is a value of f . To these sections g we can apply the known results concerning the lower bounds of binary and ternary quadratic forms. By proper choice of these sections, we derive information sufficient to prove Theorems A and B.

To avoid stretching this paper to inordinate length, I assume throughout that $L(f)$ is attained.⁸ I have shown elsewhere⁹ in the case of indefinite ternary forms how this restriction may be removed. Precisely the same type of argument will serve in the present case.

Sections 2 and 3 are devoted to the results and notions required from the theory of binary and ternary forms.

Section 4 deals with notation and the reduction of the quaternary form to a *standard* form, convenient for the analysis.

Section 5 is devoted to the proof of Theorem A. The proof of Theorem B occupies §§ 6 and 7.

2. Properties of binary quadratic forms.¹⁰ Let $q(x, y) = ax^2 + bxy + cy^2$, which we also denote by $[a, b, c]$ or $(a, \frac{1}{2}b, c)$. The hessian or determinant of q we denote by $\Delta(q)$. The discriminant is defined by

$$(2.1) \quad d = b^2 - 4ac = -4\Delta(q).$$

2.1. Definite forms. $q = [a, b, c]$ is definite if $\Delta > 0$ being positive definite if $a > 0$, negative definite if $a < 0$.

LEMMA 1. *If q is definite, then q has a non-zero minimum which is properly represented.*

We say that the positive form q is *reduced* if

$$(2.11) \quad -a < b \leq a \leq c \quad \text{and} \quad b \geq 0 \quad \text{if} \quad c = a.$$

⁸ If f is definite, the lower bound is always attained.

⁹ Dickson, 2.

¹⁰ For proofs of the results in § 2, see Dickson, 1 and 2.

We shall say that q is reduced in the wide sense if the last condition is omitted.

LEMMA 2. Let $q = [a, b, c]$ be positive definite. Let a be the minimum of q . There is a unique integer k such that the parallel transformation $x = X + kY$, $y = Y$ carries q into $Q = [A, B, C]$, where Q is reduced in the wide sense.

Plainly $A = a$, $B = 2ak + b$. There is a unique integer k such that $-a = -A < B \leq a$. Since a is the minimum of q and so of Q , we have $C \geq A = a$. The lemma follows.

LEMMA 3. Every positive definite binary form is equivalent to a reduced form $[a, b, c]$.

LEMMA 4. The least value taken by a positive reduced form $[a, b, c]$ is a . Any other value, v , of q , with the possible exceptions of $4a, 9a, \dots$, is greater than or equal to c . In particular, if v/a is not the square of an integer, then $v \geq c$. Further, in a reduced form,

$$(2.12) \quad a^2 \leq ac \leq 4\Delta/3, \quad 3ac \leq a(4c - a) \leq 4\Delta.$$

If $y \neq 0$, we have

$$ax^2 + bxy + cy^2 - c \geq ax^2 - a|xy| + a(y^2 - 1) = a(x^2 - |xy| + y^2 - 1) \geq 0.$$

If $y = 0$, the values of q are $a, 4a, 9a, \dots$. The first three statements of the lemma follow at once. The last follows from (2.01) and (2.111).

LEMMA 5. Equivalent positive reduced forms are identical.

Similar results hold for negative definite forms.

2.2. Indefinite forms. $q = [a, b, c]$ is indefinite if $d > 0$ (or $\Delta < 0$). Let $R^2 = d$, $R > 0$. We say that q is reduced if

$$(2.21) \quad 0 < R - b < 2|a| < R + b.$$

Since $R^2 - b^2 = -4ac$, these inequalities are equivalent to

$$(2.22) \quad 0 < R - b < 2|c| < R + b.$$

In a reduced form q , a and c have opposite signs.

Df. Right neighboring form. If the proper transformation $x = Y$, $y = -X + kY$ where k is an integer not zero, carries q into r , we say that r is a right hand neighbor of q .

q is called a left hand neighbor of r .

LEMMA 6. Any indefinite binary form is equivalent to a reduced form.

LEMMA 7. Every reduced form has a unique reduced right (or left) neighbor.

To any indefinite form $q = [a, b, c]$ of discriminant d corresponds a chain (g_i) , extending to infinity in both directions (if the numbers $(R+b)/2a$ are irrational), of equivalent reduced forms, each form being the right (left) hand neighbor of its predecessor (successor).

LEMMA 8. *Equivalent reduced forms belong to the same chain.*

Consequently the same chain is determined by any one of its links.

Notation for chain (g_i) . We write

$$(2.31) \quad g_i = [(-)^i A_i, B_i, (-)^{i+1} A_{i+1}], \quad A_i > 0, B_i > 0 \text{ (all } i),$$

$$(2.32) \quad B_{i-1} + B_i = 2g_{i-1} A_i, \quad g_i \text{ a positive integer,}$$

$$(2.33) \quad F_i = (g_i, g_{i+1}, \dots), \quad S_i = (0, g_{i-1}, g_{i-2}, \dots),$$

$$(2.34) \quad K_i = F_i + S_i = R/A_{i+1}, \quad F_i - S_i = B_i/A_{i+1}, \quad F_i S_i = A_i/A_{i+1},$$

$$(2.35) \quad d = R^2 = B_i^2 + 4A_i A_{i+1}, \quad R > 0.$$

The roots of $g_i(t, 1) = 0$ are $(-)^i/F_i$ and $(-)^{i+1}/S_i$. The transformation $x = Y, y = -X + k_i Y, k_i = (-)^i g_i$, carries g_i into g_{i+1} .

We suppose that the numbers $(R \pm B_i)/2A_i$ are irrational. Otherwise the chain would terminate. In other words we suppose that neither root of the quadratic in $t, q(t, 1) = 0$, is rational.

LEMMA 9. (Lagrange.) *The numbers $(-)^i A_i$ include all those numbers numerically $\leq \frac{1}{2}R$ which are properly represented by q .*

Lemma 9 follows from Lemma 8 and

LEMMA 10. *Let $q = [a, b, c]$ be indefinite. Let $|a| \leq \frac{1}{2}R$. There is a unique parallel transformation $x = X + kY, y = Y, k$ integral, which makes q reduced.*

The transformation $x = X + kY, y = Y$ carries $[a, b, c]$ into $[A, B, C]$ where $A = a, B = 2ak + b$. There is a unique integer k such that

$$0 < R - B = R - 2ak - b < 2|A| = 2|a|.$$

Since $2|A| \leq R, R - B < R, B > 0$ and therefore $2|A| \leq R < R + B$. The form $[A, B, C]$ is reduced.

Let $L(q)$ denote the lower bound of $|q|$ for all integral values of x, y , the pair $(0, 0)$ excepted. Let (g_i) be the corresponding chain.

LEMMA 11. *$L(q)$ is the lower bound of the numbers A_i , and is therefore equal to R divided by the upper bound of the numbers K_i .*

2.5. *Markoff's Theorem on the minima of indefinite binary quadratic forms.* We consider all indefinite binary quadratic forms f of discriminant $d = R^2, R > 0$.

THEOREM 1. We have the following possibilities:

$$\begin{aligned} L(f) &= \sqrt{\frac{d}{5}}, & f \sim f_0 &= \sqrt{\frac{d}{5}}(x^2 + xy - y^2); \\ L(f) &= \sqrt{\frac{d}{8}}, & f \sim f_1 &= \sqrt{\frac{d}{8}}(x^2 + 2xy - y^2); \\ L(f) &= \sqrt{\frac{25d}{221}}, & f \sim f_2 &= \sqrt{\frac{d}{221}}(5x^2 + 11xy - 5y^2); \end{aligned}$$

... The set $f_0, f_1, \dots$ continues indefinitely: each f_i has discriminant d and $L(f_i) > \frac{1}{3}R$. Each is proportional to an integral form. Further, if $L(f) > \frac{1}{3}R$, then f is equivalent properly or improperly to one of the f_i . Each f_i represents both $L(f_i)$ and $-L(f_i)$.

LEMMA 12. If the coefficients of f are incommensurable or if f does not represent both $L(f)$ and $-L(f)$, then $L(f) \leq \frac{1}{3}R$.

3. Ternary quadratic forms.

3.1. Indefinite ternary quadratic forms.

THEOREM 2. (Markoff)¹¹. Let Φ be an indefinite ternary quadratic form of non-zero hessian Δ and absolute lower bound $L(\Phi)$. We have the following possibilities:

$$(3.11) \quad L^3 = \frac{2}{3}|\Delta| \text{ and } \Phi \sim \Phi_1 = \left(\frac{2}{3}\Delta\right)^{\frac{1}{3}}(x^2 - y^2 - z^2 + xy + xz);$$

$$(3.12) \quad L^3 = \frac{2}{5}|\Delta| \text{ and } \Phi \sim \Phi_2 = \left(\frac{2}{5}\Delta\right)^{\frac{1}{3}}(x^2 - y^2 - z^2 + yz + 2zx + xy);$$

$$(3.13) \quad L^3 = \frac{1}{3}|\Delta| \text{ and } \Phi \sim \Phi_3 = \left(\frac{1}{3}\Delta\right)^{\frac{1}{3}}(x^2 - y^2 - z^2 + 2xy + 2xz);$$

$$(3.14) \quad L^3 = \frac{8}{25}|\Delta| \text{ and } \Phi \sim \Phi_4 = \left(\frac{8}{25}\Delta\right)^{\frac{1}{3}}(x^2 - y^2 - \frac{3}{2}z^2 + yz + 2zx + xy);$$

or else

$$(3.15) \quad L^3 < .30033|\Delta|.$$

LEMMA 13. If the coefficients of Φ are incommensurable or if Φ does not represent both L and $-L$, then $L^3 < .30033|\Delta|$.

For $\Phi_1, \dots, \Phi_4$ have commensurable coefficients and represent both L and $-L$.

Suppose now that Φ attains its lower bound $L(\Phi)$. We may take

$$(3.21) \quad \begin{aligned} \Phi &= ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy, \\ a\Phi &= (ax + hy + gz)^2 + \varphi(y, z), \end{aligned}$$

$$(3.22) \quad \varphi(y, z) = Cy^2 - 2Fyz + Bz^2, \quad BC - F^2 = a\Delta,$$

where $|a| = L$. Let $L > 0$. The binary form φ is negative definite if $a\Delta > 0$ and indefinite if $a\Delta < 0$. We may suppose that C and $a\Delta$ have opposite signs and that φ is reduced. By a parallel transformation on x , we can reduce (Lemmas 2 and 10) both the binaries (a, h, b) and (a, g, c) without altering φ . Reduction here may be in the wide sense.

¹¹ Markoff, 2 for Φ_1, Φ_2 and Φ_3 ; Dickson, 2 for (3.14) and (3.15). Exposition of Theorem 2 in Dickson, 2.

Under these conditions we shall say that Φ is in *standard form*. The standard form is not necessarily unique.

LEMMA 14. *Let Φ be in standard form with $a > 0$. Then, if $a^3 = \frac{2}{3}\Delta$, Φ/a is equal to*

$$(3.31) \quad (x + \frac{1}{2}y + \frac{1}{2}z)^2 - \frac{5}{4}y^2 - \frac{1}{2}yz - \frac{5}{4}z^2;$$

and, if $a^3 = -\frac{2}{3}\Delta$, Φ/a is equal to

$$(3.32) \quad (x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{3}{4}y^2 + \frac{3}{2}yz - \frac{5}{4}z^2 \quad \text{or} \quad (x + \frac{1}{2}z)^2 + y^2 + yz - \frac{5}{4}z^2.$$

We may take $a = 2$ so that $\Delta = \pm 12$. Since Φ is equivalent to Φ_1 of Theorem 2, the coefficients of which are integral multiples of $\frac{1}{2}L = \frac{1}{2}a = 1$, the coefficients $a, \dots, h$ and therefore the cofactors B, C, F of Φ are integers.

If $\Delta = 12$, φ is a reduced negative definite binary form so that

$$(3.33) \quad |2F| \leq |C|, \quad |2F| \leq |B|, \quad BC - F^2 = a\Delta = 24.$$

The indefinite binary forms (a, h, b) and (a, g, c) have minimum a . Theorem 1 yields therefore

$$(3.34) \quad -C = 5 \text{ or } -C \geq 8; \quad -B = 5 \text{ or } -B \geq 8.$$

From (3.33) and (3.34) we find $B = C = -5$, $F = 1$. Since (a, h, b) is reduced indefinite, of minimum a , we have

$$(3.35) \quad 0 < \sqrt{5} - h < a \leq -b < \sqrt{5} + h, \quad -5 = ab - h^2 = 2b - h^2,$$

whence $h = 1$, $b = -2$. Similarly $g = 1$, $c = -2$. We obtain (3.31).

If $\Delta = -12$, φ is reduced indefinite with $C > 0$ so that $B < 0$ and

$$(3.36) \quad 0 < \sqrt{12} + F < C < \sqrt{12} - F, \quad BC - F^2 = -24.$$

The indefinite binary (a, g, c) gives (3.342). The definite binary (a, h, b) of minimum a gives $C \geq 3$. Using (3.36), we obtain $C = 4$, $F = -3$, $B = -5$ or $C = 4$, $F = -2$, $B = -5$.

As before, $g = 1$, $c = -2$. Since (a, h, b) is reduced positive definite and $2b - t^2 = C = 3$ or 4 , we must have $b = 2$, $t = 1$ or $b = 2$, $t = 0$. We obtain the two cases of (3.32).

Observe that if C is the minimum of $\varphi(y, z)$ then the first case must hold.

LEMMA 15. *Let Φ be in standard form, $a > 0$. Then,*

$$(3.41) \quad \text{if } a^3 = \frac{2}{5}\Delta, \quad \Phi/a = (x + \frac{1}{2}y + z)^2 - \frac{5}{4}y^2 - 2z^2;$$

$$(3.42) \quad \text{if } a^3 = \frac{1}{8} \Delta, \quad \Phi/a = (x+y+z)^2 - 2y^2 - 2yz - 2z^2;$$

$$(3.43) \quad \text{if } a^3 = \frac{8}{25} \Delta, \quad \Phi/a = (x + \frac{1}{2}y + z)^2 - \frac{5}{4}y^2 - \frac{5}{2}z^2;$$

$$(3.51) \quad \begin{aligned} &\text{if } a^3 = -\frac{2}{5} \Delta, \quad \Phi/a \text{ is equal to} \\ &\quad (x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{3}{4}y^2 + \frac{5}{2}yz - \frac{5}{4}z^2 \\ &\quad \text{or } (x + \frac{1}{2}y + z)^2 + \frac{3}{4}y^2 + 2yz - 2z^2; \end{aligned}$$

$$(3.52) \quad \text{if } a^3 = -\frac{1}{8} \Delta, \quad \Phi/a = (x+z)^2 + y^2 + 2yz - 2z^2;$$

$$(3.53) \quad \text{if } a^3 = -\frac{8}{25} \Delta, \quad \Phi/a = (x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{5}{4}y^2 + \frac{5}{2}yz - \frac{5}{4}z^2.$$

In (3.41) and (3.43) we may interchange y and z . The proof of these results is similar to the proof of Lemma 14 and may therefore be omitted.

LEMMA 16. Let $\Delta > 0$ and suppose that Φ represents L but not $-L$. Then $L^3 \leq \frac{64}{243} \Delta$.

We may take $a = L$ and suppose that φ is reduced negative definite so that $BC \leq \frac{4}{3}a\Delta$.

The binary forms (a, h, b) , (a, g, c) have minimum a , are indefinite and do not represent $-a$. Therefore by Lemma 12, $a^2 \geq -4B^2/9$, $a^2 \leq -4C^2/9$. These inequalities yield Lemma 16.

Lemma 16 holds also if Φ does not represent numbers arbitrarily close to $-L$.

3.6. *Definite ternary quadratic forms.* Let Φ be a positive definite ternary quadratic form. We may suppose that a is the minimum of Φ and that C is the minimum of the positive definite binary form $q(y, z)$.

Then¹²

$$(3.61) \quad a^3 \leq \frac{4}{3}C, \quad aC \leq \frac{8}{2}\Delta, \quad a^3 \leq 2\Delta.$$

4. Indefinite quaternary quadratic forms. Notation and standard form. Let

$$(4.1) \quad \psi = \psi(x, y, z, t) = (a, b, c, d, f, g, h, u, v, w)(x, y, z, t)^2$$

be an indefinite quaternary quadratic form of absolute lower bound $m \geq 0$. We shall suppose that $m > 0$, so that necessarily $\Delta(\psi) \neq 0$, and that m or $-m$ is attained. Considering if necessary $-\psi$ we may therefore suppose that m is represented and plainly properly represented by ψ . We may take $d = m$ and write

$$(4.21) \quad d\psi = (ux + vy + wz + dt)^2 + \Phi(x, y, z),$$

¹² Korkine and Zolotareff, 1.

$$(4.22) \quad \begin{aligned} \Phi &= (\alpha, \beta, \gamma, \varrho, \sigma, \tau) (x, y, z)^2, \\ \alpha &= ad - u^2, \dots, \varrho = fd - vw, \dots, \end{aligned}$$

$$(4.3) \quad n = L(\Phi), \quad n > 0.^{13}$$

We suppose that n is attained¹⁴ by $|\Phi|$ and we take

$$(4.31) \quad \alpha = -n \text{ if } \Phi \text{ represents } -n,$$

$$(4.32) \quad \alpha = n \text{ if } \Phi \text{ represents } n \text{ but not } -n.$$

Write now

$$(4.33) \quad \alpha \Phi = (\alpha x + \tau y + \sigma z)^2 + d\varphi(yz), \quad \varphi = Cy^2 - 2Fyz + Bz^2,$$

where $B, C, \dots$ are the cofactors of $b, c, \dots$ in Δ . It is easily verified that

$$(4.4) \quad \Delta_1 = \Delta(\Phi) = d^2\Delta, \quad \Delta_2 = \Delta(\varphi) = BC - F^2 = \alpha\Delta.$$

A proper transformation on y and z and a parallel transformation on x puts the ternary form Φ into standard form (§ 3).

By a parallel transformation on t , which plainly does not disturb Φ , we can reduce the three binary forms obtained from Ψ by putting two of x, y, z equal to zero, in virtue of Lemmas 2 and 10 since d is evidently the minimum of any section of Ψ when that section represents d .

In the case of φ the reduction is in the strict sense. For other binary sections, when definite, the reduction must be interpreted occasionally in the wide sense.

When this reduction has been performed we shall say that Ψ is in *standard form*. The standard form is not necessarily unique.

Signature. Let S denote the signature of Ψ . Our discussion falls into two essentially distinct parts:

$$(4.5) \quad S = 0 \text{ so that } \Delta > 0, \Phi \text{ is indefinite of signature } -1:$$

$$(4.6) \quad \begin{cases} S = -2, \Delta < 0, \Phi \text{ is negative definite;} \\ S = +2, \Delta < 0, \Phi \text{ is indefinite of signature } +1. \end{cases}$$

5. The case $S = 0, \Delta > 0, \Phi$ indefinite of signature -1 .

(5.1) $\alpha = -n$. The binary (a, u, d) has positive discriminant $-4\alpha = 4n$ and minimum d so that by Theorem 1,

$$(5.12) \quad d^2 = \frac{4}{5}n \quad \text{or} \quad d^2 \leq \frac{1}{2}n.$$

¹³ If $n = 0$, then we can make $|d\Psi| < \frac{1}{4}d^2 + e$ for any $e > 0$. But $d \neq 0$ and is the lower bound of $|\Psi|$.

¹⁴ This restriction will be removed later.

Since Φ is indefinite of minimum n and hessian $d^2\Delta$ by (4.4), Theorem 2 yields

$$(5.14) \quad n^3/d^2\Delta = \frac{2}{8}, \frac{2}{5} \quad \text{or} \quad \leq \frac{1}{3}.$$

Case (5.121) with (5.141) or (5.142). Here, by Theorem 2, we may take

$$\Phi(x, y, z) = n(-x^2 - y^2 + z^2 + yz + \lambda zx + \mu xy),$$

the precise values of λ and μ being irrelevant, and so with an obvious change of notation,

$$\psi/d = (ux + vy + wz + t)^2 + \frac{5}{4}(-x^2 - y^2 + z^2 + yz + \lambda zx + \mu xy).$$

Herein take $x = 1, y = z = 0$. Since $|\psi/d| \geq 1$, $|(u + t)^2 - \frac{5}{4}| \geq 1$ for any integer t , so that $2u$ is an odd integer. We may take $2u = 1$. Similarly, by taking $x = z = 0, y = 1$, we obtain $2v = 1$. Now put $x = 0, y = -1, z = 1$. We find that $2w - 1$ must be an odd integer and so $w = 0$. But now $\psi(0, 3, 1, 1) = 0$, whereas ψ has minimum $d \neq 0$.

From (5.121) and (5.143) and from (5.122) and (5.14) we obtain respectively

$$d^6 \leq \left(\frac{1}{5}\right)^3 \frac{1}{3} d^2\Delta, \quad d^6 \leq \frac{1}{25} \frac{2}{3} d^2\Delta.$$

We conclude that, if $S = 0$ and Φ represents $-n$, then

$$(5.15) \quad d^4 \leq \frac{64}{875} \Delta.^{15}$$

(5.2) Case $\alpha = n$. Here Φ is indefinite and does not represent $-n$. We suppose that ψ is in standard form. Then q is negative definite reduced so that we may suppose

$$(5.23) \quad 0 < -C \leq -B, \quad \frac{3}{4}BC \leq -C(-B + \frac{1}{4}C) \leq \alpha\Delta \quad \text{by (2.122)}.$$

The binary forms $(\alpha, \tau, \beta), (\alpha, \sigma, \gamma)$ have Hessians dC, dB and minimum α . They do not represent $-\alpha$. By Lemma 12 therefore

$$(5.24) \quad \alpha^2 \leq -\frac{4}{9}dC, \quad \alpha^2 \leq -\frac{4}{9}dB.$$

Consider now the ternary form $\psi_{y=0}$. It has hessian $B \neq 0$ and minimum $d > 0$. Therefore it is indefinite and Theorem 2 gives

$$(5.26) \quad -d^3/B = \frac{2}{8}, \frac{2}{5}, \frac{1}{8}, \frac{8}{25} \quad \text{or} < .30033,$$

¹⁵ Improvement is possible but not worth while.

and similarly from $\Psi_{z=0}$

$$(5.27) \quad -d^3/C = \frac{2}{8}, \frac{2}{5}, \frac{1}{8}, \frac{8}{25} \quad \text{or} < .30033.$$

Number the cases of (5.26) and (5.27) $B_1, \dots, B_5, C_1, \dots, C_5$, respectively.

Case $B_5 C_5$. Employing (5.23) and (5.24) we obtain

$$(B_5 C_5) \quad d^{18} < \left(\frac{8}{25}\right)^6 \left(\frac{4}{8}\right)^3 \frac{64}{248} d^2 \Delta^4, \quad d^4 < \frac{2^7 2}{3^2 5^3} \Delta < \frac{64}{375} \Delta.$$

Case $B_5 C_i$ ($i = 1, 2, 3, 4$). Since Ψ is in standard form, the ternary section $\Psi_{y=0}$ is also in standard form. The second parts of Lemmas 14 and 15 are applicable and yield in the respective cases¹⁶

$$(5.28) \quad \alpha/d^2 = \frac{3}{4}(C_1, C_2), \quad \alpha/d^2 = 1(C_3), \quad \alpha/d^2 = \frac{5}{4}(C_4).$$

From (5.26), (5.27), (5.23) and (5.28), we obtain

$$d^4/\Delta < \frac{64}{375}(B_1, B_2), \quad d^4/\Delta < \frac{32}{225}(B_3), \quad d^4/\Delta < \frac{64}{375}(B_4)$$

and consequently

$$(B_5 C_i) \quad d^4 < \frac{64}{375} \Delta.$$

Case $B_i C_j$ ($i, j = 1, 2, 3, 4$). In addition to (5.28) we have

$$(5.29) \quad \alpha/d^2 = \frac{3}{4}(B_1, B_2), \quad \alpha/d^2 = 1(B_3), \quad \alpha/d^2 = \frac{5}{4}(B_4).$$

Inspection of (5.26) and (5.27) shows that the only possible combinations are $B_i C_j$ ($i, j = 1, 2$), $B_3 C_3$ and $B_4 C_4$. For $B_3 C_3$ and $B_4 C_4$ we have respectively

$$d^4/\Delta \leq \frac{4}{27}, \quad d^4/\Delta \leq \frac{64}{375}.$$

Case $B_1 C_1$. By the second part of Lemma 14 and the footnote above, we have

$$(5.31) \quad \Psi_{y=0} = d(t^2 + x^2 - z^2 + 2xz + zt + xt),$$

$$(5.32) \quad \Psi_{z=0} = d(t^2 + x^2 - y^2 + 2xy + yt + xt),$$

whence

$$(5.33) \quad \Psi/d = t^2 + x^2 - y^2 - z^2 + xt + yt + zt + 2xy + 2xz + 2fyz,$$

and

$$(5.34) \quad \Delta = \frac{8}{4} d^4 [4 - (f-1)^2].$$

But

$$(5.35) \quad \frac{9}{4} d^6 = BC \leq \frac{4}{3} \alpha \Delta = d^2 \Delta, \quad d^4 \leq \frac{4}{9} \Delta,$$

¹⁶ The second case in Lemma 14 is excluded since $y^2 + yz - \frac{5}{4}z^2$ represents $\frac{3}{4}$ for $y = z = 1$. But by hypothesis α is the minimum of (α, σ, γ) .

so that

$$(5.36) \quad 0 \leq f \leq 2.$$

For $x = 0$, $t = -1$, $y = z = 1$, we have $\psi/d = 2f-3$ so that either $2f-3 \geq 1$, $f \geq 2$ (and so $f = 2$) or else $2f-3 \leq -1$, $f \leq 1$.

In the latter case take $x = t = 0$, $y = z = 1$. Then $\psi/d = 2f-2$ so that $2f-2 \leq -1$, $2f \leq 1$.

Put $x = 0$, $t = y$ in ψ/d . We obtain the binary form $y^2 + (2f+1)yz - z^2$ indefinite, of minimum unity and discriminant $(2f-1)^2 + 4 \leq 8$. By Theorem 1 therefore, this discriminant must be either 8 or 5 whence $f = \frac{1}{2}$ or 0. The former value is excluded since $\psi(1, -1, 1, 1) = (1-2f)d$.

We are left with $f = 0$ and $f = 2$. The form in the latter case transforms improperly into the form from $f = 0$ when we replace z by $-z$ and x by $x+2z$. But interchange of y and z , an improper transformation, does not alter (5.33). It follows that $(B_1 C_1)$ yields the form

$$(5.37) \quad \psi_1 = d(t^2 + x^2 - y^2 - z^2 + 2zx + 2xy + xt + yt + zt), \quad d^4 = \frac{4}{9} \Delta$$

by (5.34). This is the first form in Theorem A.

It remains to show that this form actually has minimum d . But

$$4\psi_1/d = (2x+2y+2z+t)^2 + 3t^2 - 2(2y+z)^2 - 6x^2,$$

and the form $X^2 + 3T^2 - 2Y^2 - 6Z^2$ is not null, as is easily shown by descent.

Case $B_2 C_1$. We have (5.32), and by Lemma 15, $\psi_{y=0}/d$ is equal to

$$(t + \frac{1}{2}x + z)^2 + \frac{3}{4}x^2 + 2xz - 2z^2 \quad \text{or} \quad (t + \frac{1}{2}x + \frac{1}{2}z)^2 + \frac{3}{4}x^2 + \frac{5}{2}xz - \frac{5}{4}z^2$$

so that either

$$(5.42) \quad \psi/d = (t + \frac{1}{2}x + \frac{1}{2}y + z)^2 + \frac{3}{4}x^2 - \frac{5}{4}y^2 - 2z^2 + \frac{3}{2}xy + 2xz + 2\varrho yz$$

or

$$(5.43) \quad \psi/d = (t + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{3}{4}x^2 - \frac{5}{4}y^2 - \frac{5}{4}z^2 + \frac{3}{2}xy + \frac{5}{2}xz + 2\varrho yz.$$

Replace in (5.43) x by $-x-2y-3z$ and t by $-t$, a proper transformation. We obtain (5.42) with $\varrho = \frac{1}{4}$ in place of ϱ . Our transformation leaves the binary form $q = (C, -F, B)$ unaltered and so still reduced negative definite. We may thus confine ourselves to the case (5.42). Now apart from a multiplier, φ is $(24, 12-12\varrho, 40)$ so that

$$(5.44) \quad -24 < 24-24\varrho \leq 24, \quad 0 \leq \varrho < 2, \quad 1 \leq 2f < 5 \quad \text{if} \quad 2f = 2\varrho + 1.$$

Since $\psi(0, 0, 1, 1) = (2f-2)d$, we have either $2f-2 \leq -1$, $2f \leq 1$ and so $2f = 1$, or else $2f-2 \geq 1$, $2f \geq 3$. Now $\psi(0, 1, 1, -1)$

$= (2f-4)d$. If $2f-4 \geq 1$, then $2f \geq 5$ contradicting (5.44). Therefore $2f-4 \leq -1$, $2f \leq 3$ and so $2f = 3$. But $\psi(-1, 1, 1, 1) = (2f-3)d$ so that $2f \neq 3$.

The only remaining possibility, $2f = 1$, yields the form

$$(5.45) \quad \psi_2 = d(x^2 - y^2 - z^2 + t^2 + xt + yt + 2zt + 3zx + 2xy + xz),$$

the second form in Theorem A.

To show that ψ_2 has minimum d , it is sufficient to show that ψ_2/d is not null. Now $12\psi_2/d$ is equal to

$$(5.46) \quad 3(2t+x+y+2z)^2 + (3x+3y+4z)^2 - 6(2y+z)^2 - 34z^2,$$

and the form $3T^2 + X^2 - 6Y^2 - 34Z^2$ is not null. If it is null, we may suppose that $3T^2 + X^2 = 6Y^2 + 34Z^2$ where X, Y, Z, T are not all even. Since $3T^2 + X^2$ cannot be congruent to 2 or 6 modulo 8, we see that Y and Z are both even or both odd. If Y and Z are even, then $3T^2 + X^2 \equiv 0 \pmod{8}$, X and T are even, whereas X, Y, Z, T are not all even. Hence Y and Z are odd, $3T^2 + X^2 \equiv 8 \pmod{16}$ which is impossible. Our assertion follows.

Case $B_1 C_2$. We obtain $B_1 C_2$ from $B_2 C_1$ by interchanging Y and Z and so obtain (5.45) with Y and Z interchanged, an improper transformation. But from (5.46) the improper transformation $x = -X, y = -Y, z = -Z, t = T + X + Y + 2Z$ leaves ψ_2 unaltered. Thus ψ_2 is improperly equivalent to itself and the cases $B_1 C_2, B_2 C_1$ both yield the form (5.45).

Case $B_2 C_2$. Here we have

$$(5.51) \quad d^6 = \frac{4}{25} BC, \quad BC \leq \frac{4}{3} \alpha \Delta = d^2 \Delta, \quad d^4 \leq \frac{4}{25} \Delta.$$

Actually by using Lemma 15 and the arguments of $B_1 C_1, B_2 C_1$, we can show that $B_2 C_2$ leads to the form

$$(5.52) \quad \psi_3 = d[x^2 - y^2 - z^2 + t^2 + 3xy + 3xz + 4yz + xt + yt + zt], \quad d^4 = 4\Delta/33.$$

Summing up the various cases we obtain Theorem A in the case when $|\Phi|$ represents n .

(5.6) $|\Phi|$ does not represent n . Then for an infinity of values α of Φ we have either

$$(5.61) \quad -n - \epsilon n < \alpha < -n \quad \text{or} \quad n < \alpha < n + \epsilon n$$

where ϵ is an arbitrarily small positive number. If (5.61) is satisfied for arbitrarily many α 's, then Theorem 1 and Lemma 12 applied to the indefinite binary form (a, u, d) , give for some α ,

$$d^2 \leq -\left(\frac{1}{9} + \delta\right)\alpha, \quad \delta > 0.$$

Also by Theorem 2 we have certainly $n^3 < 8d^2\Delta/25$. We obtain

$$(5.62) \quad d^4 < \left(\frac{4}{9} + \delta\right)^3 \frac{8}{25} (1 + \varepsilon)^3 \Delta, \quad d^4 \leq \frac{4^3}{9^3} \cdot \frac{8}{25} \Delta,$$

since ε and δ are arbitrary.

If however (5.611) is not satisfied for an infinity of α 's, then (5.612) must hold for arbitrarily many α 's. There exists therefore an α satisfying (5.612) and such that both

$$(5.63) \quad -d^3/B < \frac{8}{25} \quad \text{and} \quad -d^3/C < \frac{8}{25}.$$

Also (corresponding to (5.24)) we have

$$(5.64) \quad n^2 \leq -\frac{4}{9}dB, \quad n^2 \leq -\frac{4}{9}dC.$$

From (5.23), (5.612), (5.63) and (5.64) we obtain

$$d^{18} < \left(\frac{8}{25}\right)^6 B^3 C^3 \leq \left(\frac{8}{25}\right)^6 \left(\frac{4}{8}\right)^3 \alpha^3 \Delta^3 < \left(\frac{8}{25}\right)^6 \left(\frac{4}{8}\right)^3 (1 + \varepsilon)^4 \left(\frac{4}{9}\right)^2 \frac{4}{8} d^3 \Delta^4,$$

for any positive ε , so that $d^4/\Delta < \frac{64}{875}$.

Summing up, we obtain Theorem A on the sole assumption that Ψ represents m . As in the case of indefinite ternary forms,¹⁷ this restriction may be removed.

6. The case $S = -2$; Φ negative definite, $\Delta < 0$. We may take $\Delta = -1$.

LEMMA 17. Ψ represents both d and $-d$ if

$$(6.01) \quad d^4 > \frac{128}{729}.$$

For if Ψ does not represent both d and $-d$, the indefinite binary form (a, a, d) of minimum d and hessian α , does not represent $-d$ so that by Lemma 12, $d^2 \leq -4\alpha/9$.

But the ternary form $-\Phi$ is definite of hessian d^2 and minimum $-\alpha$ so that by (3.613),

$$(6.02) \quad -\alpha^3 \leq 2d^2.$$

Hence $d^6 \leq \left(\frac{128}{729}\right)d^3$, $d^4 \leq \frac{128}{729}$, contradicting (6.01).

LEMMA 17 follows. Assuming (6.01), we may turn the discussion of $S = -2$ into the case $S = +2$, since $-\Psi$ has signature 2 and represents d . However, a preliminary discussion of $S = -2$ will be of service in the case $S = 2$.

¹⁷ Dickson, 2.

LEMMA 18. If $d^4 > \frac{1}{4}$, then $d^2 = -4\alpha/5$.

For if $d^2 \neq -4\alpha/5$, then $d^2 \leq -\frac{1}{2}\alpha$ by Theorem 1. From (6.02) we obtain $d^4 \leq \frac{1}{4}$.

THEOREM 3. If $S(\Psi) = -2$ and Ψ represents m , the minimum of $|\Psi|$, then $m^4 \leq -4\Delta/7$. If equality holds, then

$$\Psi \sim \Psi_1 = m(t^2 - x^2 - y^2 - z^2 + xt + yt + zt).^{18}$$

If Ψ is not equivalent to Ψ_1 , then $m^4 \leq -10\Delta/33$.

In proving Theorem 3, we may suppose that

$$(6.11) \quad d^4 = m^4 > \frac{10}{33} > \frac{1}{4},$$

so that by Lemma 18,

$$(6.12) \quad d^2 = -\frac{4}{5}\alpha.$$

The binary form $\varphi = (C, -F, B)$ is positive definite reduced so that we may take

$$(6.13) \quad 0 < C \leq B, \quad C(B - \frac{1}{4}C) \leq -\alpha = \frac{5}{4}d^2$$

by (2.122).

By Theorem 2, Lemmas 14 and 15, and (6.12) applied to the indefinite ternary forms $\Psi_{y=0}$, $\Psi_{z=0}$, we have

$$(6.14) \quad d^3/C = \frac{2}{3}, \frac{2}{5}, \frac{8}{25} \text{ or } < .30033,$$

$$(6.15) \quad d^3/B = \frac{2}{3}, \frac{2}{5}, \frac{8}{25} \text{ or } < .30033.$$

Denote the cases of (6.14), (6.15) by $C_1, \dots, C_4, B_1, \dots, B_4$.

Case $B_4 C_j$ ($j \geq 2$). Here by (6.13),

$$(6.21) \quad d^4 \leq \frac{64}{875} (j \geq 3); \quad \frac{5}{2} (\frac{25}{8} - \frac{5}{8}) d^6 < \frac{5}{4} d^2, \quad d^4 < \frac{1}{5} (j = 2).$$

Case $B_i C_1$ ($i \geq 3$). Here

$$(6.22) \quad \frac{3}{2} (\frac{25}{8} - \frac{8}{8}) d^6 \leq \frac{5}{4} d^2, \quad d^4 \leq \frac{10}{83}.$$

Case $B_1 C_1$. By Lemma 14, we have

$$(6.31) \quad \Psi_{z=0}/d = (t + \frac{1}{2}x + \frac{1}{2}y)^2 - \frac{5}{4}x^2 - \frac{1}{2}xy - \frac{5}{4}y^2,$$

and a similar expression for $\Psi_{y=0}$ on interchanging y and z . Hence

¹⁸ Ψ_1 is improperly equivalent to itself by interchange of y and z .

$$(6.32) \quad \psi/d = (t + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)^2 - \frac{5}{4}x^2 - \frac{5}{4}y^2 - \frac{5}{4}z^2 - \frac{1}{2}xy - \frac{1}{2}xz + 2\varrho yz,$$

the coefficient of yz being $2f = 2\varrho + \frac{1}{2}$.

Apart from a multiplier, the reduced binary form q is $(24, 4 - 20f, 24)$ so that

$$(6.33) \quad -24 < 8 - 40f \leq 24, \quad -2 \leq 5f < 4.$$

In (6.32) put $x = t = 0$. We obtain the form $-y^2 + 2fyz - z^2$, definite by (6.33), and of absolute minimum 1 so that

$$(6.34) \quad 3 \leq 4 - 4f^2, \quad -1 \leq 2f \leq 1.$$

Put now $x = 0$, $t = y$ in (6.32). We obtain the indefinite binary form $y^2 + (2f + 1)yz - z^2$ of minimum 1 and discriminant $(2f + 1)^2 + 4 \leq 8$ by (6.34). Theorem 1 yields therefore

$$(6.35) \quad (2f + 1)^2 + 4 = 8 \text{ or } 5, \quad f = \frac{1}{2} \text{ or } 0.$$

But $\psi(-1, 1, 1, 1) = (2f - 1)d$ which rules out $f = \frac{1}{2}$. The only possibility is $f = 0$ and we obtain the form ψ_1 of Theorem 3.

It remains to show that ψ_1 has the minimum d . Since $4\psi_1/d = 7t^2 - (2x - t)^2 - (2y - t)^2 - (2z - t)^2$ and $7t^2$ where $t \neq 0$, is not the sum of three squares, ψ_1 is not null and so has minimum d . The form ψ_1 is the first form of Theorem B.

Case B_2C_1 . In addition to (6.31) we have by Lemma 15.

$$(6.41) \quad \psi_{y=0}/d = (t + \frac{1}{2}x + z)^2 - \frac{5}{4}x^2 - 2z^2,$$

and therefore

$$(6.42) \quad \psi/d = (t + \frac{1}{2}x + \frac{1}{2}y + z)^2 - \frac{5}{4}x^2 - \frac{5}{4}y^2 - 2z^2 - \frac{1}{2}xy + 2\varrho yz,$$

the coefficient of yz being $2f = 2\varrho + 1$.

Apart from a multiplier, the reduced binary form q is $(24, 10 - 20f, 24)$ so that

$$(6.43) \quad -24 < 20 - 40f \leq 24, \quad -1 \leq 10f < 11.$$

In (6.42) put $x = t = 0$. We obtain the form $-y^2 + 2fyz - z^2$ of absolute minimum 1. If $f^2 = 1$, the form is null. If $f^2 > 1$, the form is indefinite and therefore (Theorem 1), $5 - 4f^2 - 4, 9 < 4f^2$, which contradicts (6.43). Hence $f^2 < 1$, the form is definite and

$$(6.44) \quad 3 \leq 4 - 4f^2, \quad -1 \leq 2f \leq 1.$$

Put now in (6.42) $x = 0$, $t = -y$. We obtain the form $y^3 + 2(1 - f)yz + z^2 = \psi$, of minimum 1. If $(1 - f)^2 \geq 1$, then $f \leq 0$, the form is indefinite and

$5 \leq (2-2f)^2-4$, which contradicts (6.43). Hence $(1-f)^2 < 1$, ψ is definite and

$$3 \leq 4-4(1-f)^2, \quad 1 \leq 2f \leq 3, \quad 2f \equiv 1,$$

by (6.44). Since $\Psi(-1, 1, 1, 1) = (2f-1)d$, $2f \not\equiv 1$.

Hence no forms can arise in $B_2 C_1$. The combination $B_2 C_1$ is not possible.

*Case $B_2 C_2$.*¹⁹ We have (6.41) and the result obtained by interchanging y and z so that

$$(6.51) \quad \Psi/d = (t + \frac{1}{2}x + y + z)^2 - \frac{5}{4}x^2 - 2y^2 - 2z^2 + 2\varrho yz, \quad 2f = 2\varrho + 2.$$

From the reduced definite form φ ,

$$(6.52) \quad -2 < 2-2f \leq 2, \quad 0 \leq f < 2.$$

Since $\Psi(1, 1, 1, 0) = (2f-1)d$ and $|\Psi/d| \geq 1$, we have $2f-1 \leq -1$ and so $f=0$, or else $2f-1 \geq 1$, $f \geq 1$. But $\Psi(1, 1, -1, 2) = (3-2f)d$. If $3-2f \leq -1$, then $f \geq 2$, which contradicts (6.52). Hence $3-2f \geq 1$, $f \leq 1$ and so $f=1$. But $\Psi(2, 1, 1, 0) = (2f-2)d$, which rules out $f=1$. We are left with $f=0$ and the form

$$(6.53) \quad \Psi_2 = d(t^2 - x^2 - y^2 - z^2 + xt + 2yt + 2zt + xy + xz), \quad d^4 = 4/15.$$

It remains to show that Ψ_2 has the minimum d . Now $4\Psi_2/d$ is equal to $(2t+2y+2z+x)^2-5x^2-2(2y+z)^2-6z^2$ and the form $T^2-5X^2-2Y^2-6Z^2$ is not null. For if $T^2-5X^2=2Y^2+6Z^2$, we may suppose that T, X, Y, Z are not all even. Since T^2-5X^2 cannot be congruent to 2 or 6 modulo 8, Y and Z are both even or both odd. If Y and Z are even, then $T^2-5X^2 \equiv 0 \pmod{8}$, whence T and X are even. Hence Y and Z are odd, and $T^2-5X^2 \equiv 8 \pmod{16}$, which is impossible. Our assertion follows.

*Case $B_3 C_2$.*²⁰ We have (6.41) with y and z interchanged and, by Lemma 15.

$$(6.61) \quad \Psi_{y=0}/d = (t + \frac{1}{2}x + z)^2 - \frac{5}{4}x^2 - \frac{5}{2}z^2,$$

so that

$$(6.62) \quad \Psi/d = (t + \frac{1}{2}x + y + z)^2 - \frac{5}{4}x^2 - 2y^2 - \frac{5}{2}z^2 + 2\varrho yz$$

where

$$(6.63) \quad -1 \leq \varrho < 1$$

from the negative definite reduced binary form $-\varphi$. We may plainly suppose that $\varrho \geq 0$.

¹⁹ We have immediately here $d^6 = (4Bc/25)$, $d^4 \leq \frac{4}{15} < \frac{19}{88}$. The present discussion is required in the sequel.

²⁰ We have immediately here $d^4 \leq \frac{1}{8}$ by (6.13).

Now

$$(6.64) \quad \Psi(4, -1, 2, 3) = (4 - 4\varrho)d, \quad \Psi(1, -1, 1, 2) = (\tfrac{1}{2} - 2\varrho)d.$$

Since $\Psi/d \geq 1$, (6.64) and $0 \leq \varrho \leq 1$ yield $4\varrho = 3$. But $\Psi(0, 1, 2, 0) = (4\varrho - 3)d$. The case $B_3 C_2$ cannot therefore arise.

From the results of (6.2)–(6.6) we derive Theorem 3. The discussion of § 7 will enable us to improve (6.22) and also to discuss the case $\alpha = -2d^2$. We shall in fact derive two more minima.

7. The case $S = 2$; Φ indefinite, $\Delta < 0$. We may take $\Delta = -1$.

LEMMA 19. *If $d^4 > \frac{64}{375}$, then Φ cannot represent $-n$.*

Let Φ represent $-n$. Then $\alpha = -n$. Theorem 1 applied to the indefinite binary form (u, u, d) of minimum d and hessian α yields

$$(7.01) \quad d^3 = \tfrac{4}{5}n \quad \text{or} \quad d^2 \leq \tfrac{1}{2}n.$$

Theorem 2 applied to the indefinite ternary form Φ of minimum n and hessian $-d^2$ yields

$$(7.02) \quad n^3/d^2 = \tfrac{2}{3}, \tfrac{2}{5} \quad \text{or} \quad \leq \tfrac{1}{3}.$$

If (7.011) and (7.021) or (7.022), then, by Theorem 2, we may take

$$(7.03) \quad \Psi/d = (ux + vy + wz + t)^2 + \tfrac{5}{4}(-x^2 + y^2 + z^2 - xy - \epsilon xz - \delta yz),$$

the precise values of ϵ and δ being irrelevant.

We observe that $|\Psi/\alpha| \geq 1$. Take $x = 1, y = z = 0$. We see that $2u$ is an odd integer. Take $x = y = 1, z = 0$. We see that $2v - 2u$ is an odd integer. But now Ψ vanishes for $x = 3, y = -1, z = 0, t = v - 3u + \tfrac{5}{2}$, which is an integer. The cases under discussion cannot therefore arise.

The remaining cases of (7.01) and (7.02) yield

$$d^6 \leq \tfrac{1}{8} \cdot \tfrac{2}{8} d^3; \quad d^6 \leq (\tfrac{4}{5})^3 \tfrac{1}{8} d^3; \quad d^4 \leq \tfrac{64}{375},$$

contradicting the hypothesis of the lemma.

Lemma 19 may be much improved, but the improvement does not serve us.

Let now $\alpha = n$. Then Φ does not represent $-n$, and Theorem 2 gives

$$(7.11) \quad \alpha^3 = \nu d^3, \quad \nu < .30033.$$

The form q is now indefinite reduced. We may identify it with any member, q_{2i} , of the chain of equivalent reduced forms to which it belongs. In the notation of § 3, we have

$$(7.12) \quad C = A_{2i} > 0, \quad -B = A_{2i+1} > 0, \quad R = 2\sqrt{\alpha}.$$

The form (α, τ, β) is definite, of minimum α and hessian dC . The form (α, σ, γ) is indefinite, of minimum α and hessian dB , and does not represent $-\alpha$. Hence

$$(7.13) \quad \alpha^2 \leq \frac{4}{3} d A_{2i}; \quad \alpha^2 \leq \frac{4}{9} d A_{2i+1} \quad (\text{all } i).$$

The ternary form $\Psi_{z=0}$ is definite and satisfies the conditions of (3.6). Hence

$$(7.14) \quad d^3 = \mu \alpha, \quad \mu \leq \frac{4}{3}; \quad \alpha d \leq \frac{3}{2} A_{2i} \quad (\text{all } i).$$

For the indefinite ternary form $\Psi_{y=0}$ of minimum d and hessian $-A_{2i+1}$, Theorem 2 yields the following possibilities:

$$(7.15) \quad d^3/A_{2i+1} = \lambda_i, \quad \lambda_i = \frac{2}{3}, \frac{2}{5}, \frac{1}{8}, \frac{8}{25} \text{ or } < .30033.$$

From (7.11) and (7.141) we deduce

$$(7.16) \quad d^4 = \mu^3 \nu \quad (\mu \leq \frac{4}{3}, \nu < .30033).$$

Since $K_i = R/A_{i+1}$, (7.14) gives

$$(7.17) \quad d^4 \leq 9\mu/K_{2i-1}^2 \quad (\text{all } i),$$

while (7.141) and (7.15) give

$$(7.18) \quad d^4 \leq 4\lambda_i^2/\mu K_{2i}^2 \quad (\text{all } i).$$

From (7.15) and (7.132)

$$(7.19) \quad d^4 \leq 8\lambda_i^{\frac{3}{2}}/3 K_{2i}^2 \quad (\text{all } i).$$

We observe that μ and ν are independent of i , while λ_i depends on i .

Let us note that for

$$(7.22) \quad \lambda = \lambda_i = \frac{2}{3}, \frac{2}{5}, \frac{1}{8} \text{ or } \frac{8}{25},$$

we have by Lemmas 14 and 15 in the respective cases,

$$(7.25) \quad \mu = d^2/\alpha = \frac{4}{3}, \frac{4}{5}, 1 \text{ or } \frac{4}{5}.$$

If therefore, for some i , $\lambda_i = \frac{8}{25}$, then $\mu = \frac{4}{5}$ and (7.16) gives

$$(7.24) \quad d^4 < \frac{64}{125} \cdot \frac{8}{25} < \frac{1}{5}.$$

We assume from now on that

$$(7.25) \quad d^4 \geq \frac{1}{5} > \frac{64}{375}.$$

From (7.22)–(7.25) we obtain the three cases below:

- I. $\lambda_i := \frac{1}{8}$ for some i ; $\mu = 1$;
 II. $\lambda_i = \frac{2}{8}$ or $\frac{2}{5}$ for some i ; $\mu = \frac{4}{8}$;
 III. $\lambda_i < .30033$ for all i ; $\mu \leq \frac{4}{8}$.

These cases are exhaustive and do not overlap. We observe that in I. we cannot have $\lambda_i = \frac{2}{8}$ or $\frac{2}{5}$ nor, in II. $\lambda_i = \frac{1}{8}$.

(7.3) *Case I.* divides into

- I₁. For some j , $\lambda_j < .30033$;
 I₂. For all j , $\lambda_j = \frac{1}{8}$.

Case I₁. From (7.17) and (7.25),

$$(7.311) \quad K_{2i-1} \leq 3\sqrt{5} < 7 \quad (\text{all } i).$$

From (7.18), (7.25)

$$(7.312) \quad K_{2i} \leq \frac{2}{8}\sqrt{5} \quad (\lambda_i = \frac{1}{8}), \quad K_{2j} < 2\lambda_j\sqrt{5} < 1 + \frac{1}{5} + \frac{1}{6}.$$

From (7.312) and (7.311) we deduce

$$(7.313) \quad g_{2i} = 1 \quad (\text{all } i), \quad g_{2i-1} \leq 5 \quad (\text{all } i),$$

since $K_i > g_i$ and $(6, 1, \dots) + (0, 1, \dots) > (6, 2) + (0, 2) = 7$.

The useful inequality

$$(7.32) \quad K_i > g_i + \frac{1}{1+g_{i-1}} + \frac{1}{1+g_{i+1}} \quad (\text{all } i)$$

applied to (7.3122) yields, in virtue of (7.313), $g_{2j-1} = g_{2j+1} = 5$.

But now $K_{2j-1} > (5, 2) + (0, 2) = 6$ and (7.131) gives, on squaring,

$$(7.321) \quad \alpha^3 < \frac{64}{9} \cdot \frac{d^2}{86}, \quad \nu < \frac{16}{81} < \frac{1}{5}, \quad d^4 = \mu^3 \nu < \frac{1}{5},$$

from (7.16), which contradicts (7.25).

Remains therefore only

$$\text{I}_2. \quad \lambda_i = \frac{1}{8}, \quad K_{2i} = K_0 \quad (\text{all } i).$$

LEMMA 20. *If the K_{2i} are all equal, then*

$$(7.33) \quad g_{2i} = g_0, \quad g_{4i+1} = g_1, \quad g_{4i+3} = g_3 \quad (\text{all } i).$$

Write $X = (g_0, g_{-1}, g_{-2}, \dots)$, $Y = (g_2, g_3, \dots)$. The equation $F_0 + S_0 = K_0 = K_2 = F_2 + S_2$ may be written

$$(0, g_1, Y) + X = (0, g_1, X) + Y.$$

Since X and Y are positive, we deduce $X = Y$ and so

$$g_0 = g_2, \quad g_{-1} = g_3, \quad g_{-2} = g_4, \dots$$

The lemma follows since K_0 and K_2 may be any two consecutive K_{2i} , K_{2i+2} .

In I_2 , therefore, the sequence of g 's is $(1^*, g_1, 1, g_3^*)$. An easy calculation gives

$$(7.34) \quad \varphi_0 = 2\sqrt{\alpha/\xi}(A_0 y^2 + B_0 yz - A_1 z^2)$$

where

$$(7.341) \quad \begin{aligned} \xi &= (g_1 g_3 + 2g_1 + 2g_3 + 2)^2 - 4 \\ &= (g_1 + 2)(g_3 + 2)(g_1 g_3 + 2g_1 + 2g_3), \end{aligned}$$

$$(7.342) \quad A_0 = g_1 + 2, \quad B_0 = g_3(g_1 + 2), \quad A_1 = g_1 g_3 + g_1 + g_3,$$

and

$$(7.343) \quad K_0^2 = \xi/A_1^2, \quad K_{-1}^2 = \xi/A_0^2.$$

We may suppose that $1 \leq g_1 \leq g_3 \leq 5$ (7.313). If $g_1 = 1$, then (7.32) gives $K_0 > 1 + \frac{1}{2} > \frac{2}{3}\sqrt{5}$, contradicting (7.3121). Thus $g_1 \geq 2$. If $g_3 = 5$, then $K_3 > 6$ and we obtain (7.321). Thus $g_3 \leq 4$.

Now from (7.18), $d^4 = (4/9K_0^2)$. The condition $d^4 \geq \frac{1}{5}$ leaves us the two cases

$$(7.35) \quad d^4 = \frac{2}{9} (g_1 = g_3 = 4), \quad d^4 = \frac{1}{5} \cdot \frac{361}{851} (g_1 = 3, g_3 = 4).$$

In (7.351) we find from (7.34) and $\mu = 1$ that $\alpha = d^2 = \sqrt{2}/3$, and

$$\Phi/\alpha = (r + r'y + \sigma'z)^2 + \frac{3}{4}(y^2 + 4yz - 4z^2),$$

the form Φ/α having minimum 1. Take $y = 0, z = 1$. Then $|(x + \sigma')^2 - 3| \geq 1$, whence σ' is an integer. Taking $z = 0, y = 1$, we see that $2r'$ is an odd integer. We may take $\sigma' = 0, r' = \frac{1}{2}$, and we obtain

$$\Psi/d = (t + ux + vy + wz)^2 + (x + \frac{1}{2}y)^2 + \frac{3}{4}(y^2 + 4yz - 4z^2), \quad |\Psi/d| \geq 1.$$

We find that w is an integer by taking $x = y = 0, z = 1$; that u is an integer by taking $y = 0, x = z = 1$; and that v is an integer by taking $x = 2, y = 1, z = 2$. We may take u, v, w all zero. Replacing y by $y - 2z$ and x by $x + z$, we obtain the equivalent form

$$(7.36) \quad \psi_3 = d(t^2 - 6z^2 + x^2 + xy + y^2), \quad d^4 = \frac{2}{9}.$$

This form has minimum d . For $4\psi_3/d$ is equal to

$$(2t)^2 - 6(2z)^2 + (2x + y)^2 + 3y^2$$

which cannot be zero, by congruential arguments, unless x, y, z, t are all zero.

Further, ψ_3 is improperly equivalent to itself, since it is unaltered by interchange of x and y .

In (7.352) however we obtain $\alpha = d^2 = (19/3\sqrt{195})$,

$$\Phi/\alpha = (x + \tau'y + \sigma'z)^2 + \frac{3}{19}(5y^2 + 20yz - 19z^2), \quad |\Phi/\alpha| \geq 1.$$

Take $y = 0, z = 1$. We obtain σ' an integer, $\sigma' = 0$. Take $y = -4, z = 1$. We see that $4\tau'$ is an integer. We may suppose that $-\frac{1}{2} < \tau' \leq \frac{1}{2}$. Take $x = 0, z = 0, y = 1$. Then $\tau'^2 + \frac{15}{19} \geq 1$. Hence $\tau' = \frac{1}{2}$. But for $x = 2, y = 1, z = -1$, we have

$$0 < \Phi/\alpha = \frac{25}{4} - \frac{102}{19} < 1,$$

a contradiction. Thus (7.352) cannot arise.

Hence (7.25) in Case I. leads only to the form ψ_3 of (7.36) with $d^4 = \frac{2}{9}$.

(7.4) *Case II.* Here $\mu = \frac{4}{8}, \alpha = 3d^2/4$. From (7.17) and (7.25)

$$(7.41) \quad K_{2i-1} \leq \sqrt{60} < 7.75, \quad g_{2i-1} \leq 7 \quad (\text{all } i).$$

But now (7.32) gives $K_{2j} > 1 + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{4}$, and (7.18) yields $d^4 < 48\lambda_j^2/25 < \frac{1}{5}$ if $\lambda_j < 15/12$. Since $15/12 = .30033$, we have in Case II.

$$(7.42) \quad \lambda_i = \frac{2}{8} \text{ or } \frac{2}{5} \quad (\text{all } i).$$

The discussion of (7.42) falls into two parts:

$$\text{II}_1. \quad \lambda_i = \frac{2}{8} \quad (\text{all } i),$$

$$\text{II}_2. \quad \lambda_i = \frac{2}{5} \quad \text{for some } i.$$

(7.5) *Case II}_1*. From (7.18) and (7.25)

$$(7.51) \quad d^4 = (4/3K_{2i}^2), \quad K_{2i}^2 < 20/3, \quad K_{2i} < \frac{2}{3}\sqrt{15} < \frac{8}{3}, \quad g_{2i} = 1 \text{ or } 2.$$

By Lemma 20,

$$(7.52) \quad g_{2i} = 1 \quad (\text{all } i) \quad \text{or} \quad g_{2i} = 2 \quad (\text{all } i).$$

The sequence of g 's is (g_0^*, g_1, g_0, g_3^*) . If $g_0 = 1$, then (7.34)–(7.343) give with (7.51)

$$(7.53) \quad \frac{3}{4} d^4 = (g_3 g_1 + g_3 + g_1)^2 / \{(g_3 + 2)(g_1 + 2)(g_3 g_1 + 2g_3 + 2g_1)\},$$

and we may suppose that $g_1 \leq g_3$.

From (7.17), (7.18), (7.343) we obtain

$$(7.54) \quad A_1 \leq 3A_0, \quad g_1 g_3 + g_1 + g_3 \leq 3g_1 + 6$$

by (7.342). *A fortiori* $g_1^2 + 2g_1 \leq 3g_1 + 6$, $g_1 \leq 3$; and (7.54) gives

$$(7.55) \quad g_1 = 1, g_3 \leq 4; \quad g_1 = 2 \leq g_3 \leq 3; \quad g_1 = 3 = g_3.$$

From (7.53) we obtain the following set of values for d^4 ,

$$g_1 = 1; \quad \frac{4}{15} (g_3 = 1), \quad \frac{25}{72} (g_3 = 2), \quad \frac{4}{3} \cdot \frac{49}{165} (g_3 = 3), \quad \frac{3}{7} (g_3 = 4):$$

$$g_1 = 2; \quad \frac{4}{9} (g_3 = 2), \quad \frac{121}{240} (g_3 = 3):$$

$$g_1 = 3; \quad \frac{4}{7} (g_3 = 3).$$

With the exception of the first and last of these numbers, all these values of d^4 exceed $\frac{10}{33}$ and are less than $\frac{4}{7}$. Also in each of these cases Ψ represents $-d$. Hence, in each case, $-\Psi$ has signature -2 , represents d and has

$$\frac{10}{33} < d^4 < \frac{4}{7}.$$

But this contradicts Theorem 3.

In the case $g_1 = g_3 = 3$, $d^4 = \frac{4}{7}$, it is easily verified that we arrive at the form

$$d[(t + \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z)^2 + \frac{3}{4}(x + \frac{1}{3}y + z)^2 + \frac{2}{3}(y^2 + 3yz - 3z^2)],$$

which is equivalent to $-\Psi_1$ of Theorem 3.

In the case of $g_1 = g_3 = 1$, $d^4 = \frac{4}{15}$, we arrive at the form

$$\Psi'_2 = d[(t + \frac{1}{2}x + \frac{1}{2}z)^2 + \frac{3}{4}(x + z)^2 + 2(y^2 + yz - z^2)]$$

which is *not* equivalent to $-\Psi_2$ of (6.53). For plainly,

$$\Psi'_2/d \sim t^2 + tx + x^2 + 2(y^2 + yz - z^2)$$

and the adjoint of $2\Psi'_2/d$ is

$$-20(2t^2 - 2tx + 2x^2) + 3(-4y^2 - 4yz + 4z^2) \equiv -10(2x - t)^2 \pmod{3}$$

and is therefore never congruent to -1 modulo 3 but is congruent to 0 or $+1$.

On the other hand, replacing t by $t-y-z$ in (6.53) we see that

$$-\Psi_2/d \sim -t^2 - tx + x^2 + 2(y^2 + yz + z^2),$$

and the adjoint of $-2\Psi_2/d$ is

$$12(t^2 + tx - x^2) - 5(4y^2 - 4yz + 4z^2) \equiv -5(2z - y)^2 \pmod{3}$$

and therefore never congruent to -1 modulo 3 but is congruent to 0 or $+1$.

It follows that the adjoints of Ψ_2 and $-\Psi_2$ are not equivalent, whence Ψ_2' and $-\Psi_2$ cannot be equivalent. They are the forms in (B 3) of Theorem B.

If $g_{2i} = 2$, we find that

$$(7.561) \quad 12d^4 = (2g_1g_3 + g_3 + g_1)^2 / \{(g_1 + 1)(g_3 + 1)(g_1g_3 + g_1 + g_3)\}, \quad \alpha = \frac{8}{4}d^2,$$

$$(7.562) \quad \Phi/\alpha = (x + \tau'y + \sigma'z)^2 + \{8/3(2g_1g_3 + g_1 + g_3) \\ \times \{4(g_1 + 1)y^2 + 4g_3(g_1 + 1)yz - (2g_1g_3 + g_1 + g_3)z^2\},$$

$$(7.563) \quad \Psi/d = (t + u'x + v'y + w'z)^2 + (3\Phi/4\alpha).$$

The first condition of (7.54) gives

$$(7.564) \quad 2g_1g_3 + g_1 + g_3 \leq 12(g_1 + 1).$$

We may suppose that $g_1 \leq g_3$. The condition $d^4 \geq \frac{1}{5}$ rules out $g_1 = 1$ and $g_1 = 2$, $g_3 \leq 5$.

From (7.564) we have therefore the following possibilities.

$$(7.565) \quad g_1 = 2, \quad g_3 = 6; \quad 3 \leq g_1 \leq g_3 \leq 6.$$

In (7.563) put $y = 0$. We obtain an indefinite ternary form of minimum 1 and hessian $-\frac{8}{2}$. By Theorem 2, therefore, $2u', u'^2 + \frac{3}{4}, 2w'u', w' + \frac{8}{2}\sigma'$ are integers, whence we may take $u' = \frac{1}{2}$, $w' = 0$, $\sigma' = 0$ by means of appropriate transformations of the type $t = t' + \lambda x' + \mu z$, $x = x' + \nu z$.

In (7.563) put $y = -g_3z$. We obtain an indefinite ternary form of minimum 1 and hessian $-\frac{8}{2}$. We find that $2v'g_3$ and $\tau'g_3$ are integers, both even or both odd.

Similarly, taking $y = -g_1Y$, $z = -(1 + 2g_1)Y$, we obtain an indefinite ternary form of minimum 1 and hessian $-\frac{3}{2}$, whence $2v'g_1$ and $\tau'g_1$ are integers, both even or both odd.

Let δ be the greatest common divisor of g_1 and g_3 . Then $2v'\delta$ and $\tau'\delta$ are both integers.

The only case in (7.565) which leads to a form with the properties desired is given by $g_1 = g_3 = 4$ and the form arising is equivalent to the form Ψ_3 of (7.36). The other cases of (7.565) lead to contradictions.

The case $g_1 = g_3 = 4$. Here

$$\Phi/\alpha = (x + \tau'y)^2 + \frac{4}{8}(y^2 + 4yz - 2z^2)$$

and $4\tau'$ is an integer. We may suppose that $\tau' = 0, \pm\frac{1}{4}$ or $\frac{1}{2}$. Take $y = 1, z = -1$. Take $x = 2$ if $\tau' = \frac{1}{2}$; $x = \mp 3$ if $\tau' = \pm\frac{1}{4}$. We obtain a number numerically less than 1, whereas $|\Phi/\alpha| \geq 1$. Hence $\tau' = 0$ and

$$\Psi/d = (t + \frac{1}{2}x + v'y)^2 + \frac{3}{4}x^2 + y^2 + 4yz - 2z^2, \quad v' = 0, \pm\frac{1}{4} \text{ or } \frac{1}{2}.$$

Take $x = 2X, y = X, z = -X$. We obtain the indefinite binary form $(t + X + v'X)^2 - 2X^2$ of minimum 1 and discriminant 8, whence (Theorem 1) v' is an integer and so $v' = 0$. We obtain the form

$$\Psi_3 = d(t^2 + tx + x^2 + y^2 + 4yz - 2z^2), \quad d^4 = \frac{2}{9},$$

which is equivalent to Ψ_3 of (7.36) on replacing y by $y - 2z$.

The remaining cases of (7.565). We treat in detail the case $g_1 = 3, g_3 = 6$, the analysis in the other cases being of the same nature. For $g_1 = 3, g_3 = 6$, we obtain

$$\Phi/\alpha = (x + \tau'y)^2 + \frac{8}{63}(16y^2 + 96yz - 21z^2), \quad \tau' = 0, \pm\frac{1}{3}.$$

Take $y = 1, z = -1$. If $\tau' = \pm\frac{1}{3}$ take $x = \mp 4$. We obtain $|\Phi/\alpha| < 1$ whereas $|\Phi/\alpha| \geq 1$. Hence $\tau' = 0$ and so

$$\Psi/d = (t + \frac{1}{2}x + v'y)^2 + \frac{3}{4}x^2 + \frac{2}{21}(16y^2 + 96yz - 21z^2), \quad v' = 0, \pm\frac{1}{3}.$$

If $v' = \pm\frac{1}{3}$, take $x = 2, y = 1, z = -1$ and $t = -4$ or 2 respectively. We obtain $|\Psi/d| < 1$ whereas $|\Psi/d| \geq 1$. Hence $v' = 0$, which is ruled out by $\Psi(0, 1, -1, 3) = -\frac{2}{3}$.

The discussion of Case II₁ is complete.

(7.6) Case II₂. For some $i, \lambda_i = \frac{2}{5}, d^3 = -\frac{2}{5}B$. By a proper transformation on x, z, t we can, by Theorem 2, take

$$\Psi_{y=0} = d(x^2 - z^2 + t^2 - xt - 2zx - zt).$$

Take now, in Ψ , z in place of t as the leading variable and write

$$c\Psi = ()^2 + \Phi_1(x, y, t), \quad c = -d.$$

The form Φ_1 is negative definite, for the signature of $c\Psi$ is -2 . The coefficient of x^2 in Φ_1 is $\alpha_1 = -5d^2/4$ and $-\alpha_1$ is the minimum of $-\Phi_1$. For if the minimum of $-\Phi_1$ is $n_1 < 5d^2/4$ we can make $|c\Psi| < d^2$ whereas $|\Psi| \geq d$.

We obtain now, as in § 6, from Φ_1 the negative definite binary form

$$\varphi(y, t) = Dy^2 - 2Vyt + Bt^2.$$

Let us observe that we can find indefinite ternary sections of Ψ having minimum d and for hessian *any* value represented by φ_1 .

Let $(D', -V', B')$ be the equivalent reduced form to φ with $0 < -B' < -D'$. Then $-B' \leq -B$.

If $-B > -B'$, then $d^3 > -\frac{2}{5}B'$. By the preceding paragraph and Theorem 2 we must have $d^3 = -\frac{2}{3}B'$. But now $B = \frac{5}{3}B'$ so that, by Lemma 4, $-B \geq -D'$. By the preceding paragraph and Theorem 2 we have either $d^3 = -\frac{2}{3}D'$ or $d^3 = -\frac{2}{5}D'$. The first case falls under B_1C_1 of § 6 and leads to the form $-\Psi_1$ of Theorem 3. The second case falls under B_2C_1 of § 6, there shown to be impossible.

Remains the case $B = B'$, so that $-B \leq -D'$,

$$d^3 = -\frac{2}{5}B'; d^3 = -\frac{2}{5}D' \text{ or } -\frac{8}{25}D' \text{ or } < -\cdot 30033 D',$$

by Theorem 2, the value $d^3 = -\frac{1}{3}D'$ being incompatible with $\alpha_1 = -5d^2/4$.

The first case leads to B_2C_3 and the form $-\Psi_2$ of § 6. The second case falls under B_3C_2 of § 6 and is therefore ruled out. Finally the third case falls under B_4C_2 of § 6 and leads to $d^4 = \frac{1}{5}$. The discussion of Π_2 is complete.

(7.7) Case III. For all i ,

$$(7.71) \quad \lambda_i < \cdot 30033, \quad \mu \leq \frac{4}{3}.$$

From (7.16) and (7.25),

$$(7.72) \quad \mu^3\nu \geq \frac{1}{5}, \quad \nu < \cdot 30033.$$

From (7.18), (7.25) and (7.72),

$$(7.73) \quad K_{2i}^2 \leq 20\lambda_i^2/\mu < 4, \quad g_{2i} = 1 \quad (\text{all } i).$$

From (7.17) and (7.25),

$$(7.74) \quad K_{2i-1}^2 \leq 60, \quad g_{2i-1} \leq 7 \quad (\text{all } i).$$

If $g_{2i-1} = 7$ for some i , then (7.32) gives $K_{2i-1} > 8$, contradicting (7.73). Hence $g_{2i-1} \leq 6$ (all i).

If $g_{2i-1} = 6$ for some i , then $K_{2i-1} > 7$, and (7.13) yields

$$(7.75) \quad \alpha^8 \leq \frac{64}{9} \frac{d^2}{K_{2i-1}^2}, \quad \nu \leq \frac{64}{441},$$

whence a stronger inequality for μ in (7.72). But now (7.731) gives

$$K_{2i} < 1.28 < 1 + \frac{1}{7} + \frac{1}{7}.$$

But by (7.32) since $g_{2i-1} \leq 6$ (all i), $K_{2i} > 1 + \frac{1}{7} + \frac{1}{7}$. Hence $g_{2i-1} \leq 5$ for all i .

If $g_{2i-1} = 5$ for some i , then $K_{2i-1} > 6$ and (7.751) yields $\nu < \frac{16}{81}$; and (7.72) gives a stronger inequality for μ .

From (7.73) and (7.72) we deduce therefore

$$(7.76) \quad K_{2i} < 3/\sqrt{5} < 1 + \frac{1}{5} + \frac{1}{6}.$$

The conditions $g_{2i-1} \leq 5$ (all i), (7.32) and (7.76) show that $g_{2i-1} = 5$ for all i . But now $K_{2i} = 3/\sqrt{5}$ contradicting (7.76). Hence $g_{2i-1} \leq 4$ for all i .

Now (7.72) and (7.731) give

$$(7.77) \quad K_{2i} < 1 + \frac{1}{4} + \frac{1}{5}.$$

If for some i , $g_{2i-1} \leq 3$, then (7.32) and $g_{2j-1} \leq 4$ (all j) yield $K_{2i} > 1 + \frac{1}{4} + \frac{1}{5}$, contradicting (7.77). Hence $g_{2i-1} = 4$ for all i . We obtain

$$(7.78) \quad K_{2i} = (1, 4, \dots) + (0, 4, 1, \dots) = \sqrt{2}, \quad K_{2i-1} = 4\sqrt{2} \quad (\text{all } i).$$

From (7.75) and (7.781)

$$\alpha^8 \leq \frac{64}{9} \frac{d^2}{32}, \quad \nu \leq \frac{2}{9}.$$

This value of ν increases the value of μ in (7.72) and we deduce from (7.731) that $K_{2i} < 1.4$, which contradicts (7.781).

Thus, if $d^4 \geq \frac{1}{5}$, no forms arise in Case III.

Summing up the results of §§ 6 and 7, we obtain Theorem B, provided that $|\Psi|$ represents m and that $|\Phi|$ represents n . The case " $|\Phi|$ does not represent n " may be discussed as in (5.6).

Finally we may remove the restriction that $|\Psi|$ represents m as in the case of indefinite ternary forms²¹.

²¹ Dickson, 2.

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